## § 4 Topological Terminologies and Results of $\mathbb{R}^n$

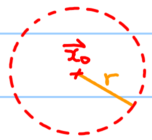
**Definition 4.1**

Let $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$.

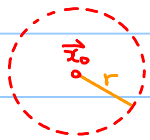
$B_r(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{x}_0| < r\}$ is said to be the open ball centered at $\vec{x}_0$ with radius $r$.

$B_r^{\circ}(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : 0 < |\vec{x} - \vec{x}_0| < r\}$ is said to be the deleted open ball centered at $\vec{x}_0$ with radius $r$.

(In fact, $B_r^{\circ}(\vec{x}_0) = B_r(\vec{x}_0) \setminus \{\vec{x}_0\}$.)




$B_r(\vec{x}_0)$

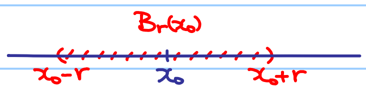


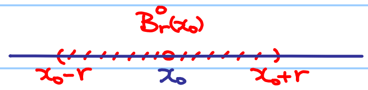

$B_r^{\circ}(\vec{x}_0)$

In particular, $n = 1$, let $x_0 \in \mathbb{R}$ and $r > 0$. Then,

$B_r(x_0) = (x_0 - r, x_0 + r)$

$B_r(x_0) = (x_0 - r, x_0) \cup (x_0, x_0 + r)$




**Definition 4.2**

Let $S \subseteq \mathbb{R}^n$.

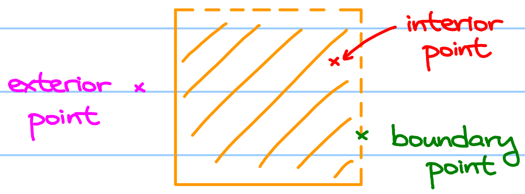
(1) $\vec{x}_0$ is said to be an interior point of $S$ if there exists $r > 0$ such that $B_r(\vec{x}_0) \subseteq S$.
  The interior of $S$ is defined as the set of all interior points of $S$, denoted by $\mathrm{Int}(S)$.

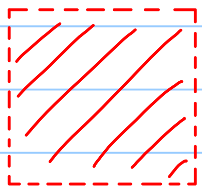
(2) $\vec{x}_0$ is said to be an exterior point of $S$ if there exists $r > 0$ such that $B_r(\vec{x}_0) \subseteq \mathbb{R}^n \setminus S$.
  The exterior of $S$ is defined as the set of all exterior points of $S$, denoted by $\mathrm{Ext}(S)$.
  (Remark: $\mathrm{Ext}(S) = \mathrm{Int}(\mathbb{R}^n \setminus S)$.)

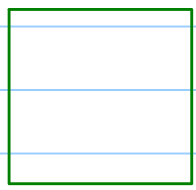
(3) $\vec{x}_0$ is said to be a boundary point of $S$ if for all $r > 0$, $B_r(\vec{x}_0) \cap S \neq \phi$ and $B_r(\vec{x}_0) \cap (\mathbb{R}^n \setminus S) \neq \phi$.
  The boundary of $S$ is defined as the set of all boundary points of $S$, denoted by $\partial S$.

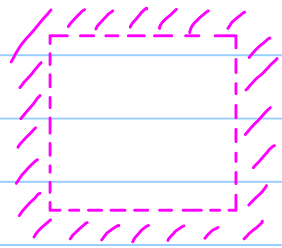
**Example 4.1**

Let $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 1\}$







$S$ $\quad$ $\mathrm{Int}(S)$ $\quad$ $\partial S$ $\quad$ $\mathrm{Ext}(S)$

Exercise 4.1

Let $S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.

Show that $\text{Int}(S) = \phi$, $\partial S = S$ and $\text{Ext}(S) = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \neq 0\}$.

Proposition 4.1

(1) $\mathbb{R}^n = \text{Int}(S) \amalg \partial S \amalg \text{Ext}(S)$, where $\amalg$ is the disjoint union.

(2) $\text{Int}(S) \subseteq S$ and $\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$.

Definition 4.3

Let $S \subseteq \mathbb{R}^n$.

(1) $S$ is open if for all $\vec{x} \in S$, there exists $r > 0$ such that $B_r(\vec{x}) \subseteq S$, i.e. $\vec{x}$ is an interior point of $S$.

(2) $S$ is closed if $\mathbb{R}^n \setminus S$ is open.

(3) $\overline{S} = \text{Int}(S) \amalg \partial S$ is said to be the closure of $S$.

Remark:

(1) Note that $\text{Int}(S) \subseteq S$. If $S$ is open, we have $\vec{x} \in S \Rightarrow \vec{x} \in \text{int}(S)$, i.e. $S \subseteq \text{int}(S)$.

Therefore $S$ is open if and only if $S = \text{int}(S)$.

(2) Note that $\text{Ext}(S) = \text{Int}(\mathbb{R}^n \setminus S)$. $\mathbb{R}^n \setminus S$ is open $\Leftrightarrow$ $\text{Ext}(S) = \text{Int}(\mathbb{R}^n \setminus S) = \mathbb{R}^n \setminus S$

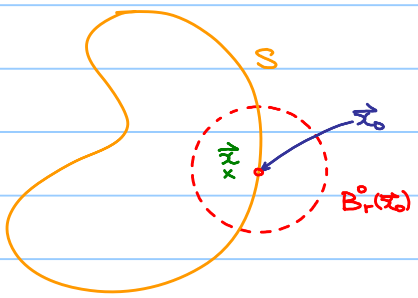
Also, $\mathbb{R}^n = \text{Int}(S) \amalg \partial S \amalg \text{Ext}(S)$.

Therefore $S$ is closed if and only if $S = \text{Int}(S) \amalg \partial S = \overline{S}$.

Definition 4.4

Let $S \subseteq \mathbb{R}^n$.

$\vec{x}_0 \in \mathbb{R}^n$ is said to be a cluster point of $S$ if for all $r > 0$, $B_r^\circ(\vec{x}_0) \cap S \neq \phi$.




No matter how small $r$ is,

there exists $\vec{x} \in S$ such that

$\vec{x} \neq \vec{x}_0$ and distance between $\vec{x}$ and $\vec{x}_0 < r$

Example 4.2

Let $S = \{-1\} \cup (0,1) \subseteq \mathbb{R}$


The set of all cluster points of $S = [0,1]$

(Even $-1 \in S$, but $-1$ is NOT a cluster point of $S$!)

Exercise 4.2

Let $S \subseteq \mathbb{R}^n$. Show that every interior point is a cluster point of $S$

Hence, if $S$ is open, then every point in $S$ is a cluster point of itself.

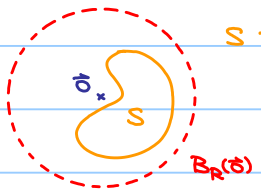
Definition 4.5

Let $S \subseteq \mathbb{R}^n$

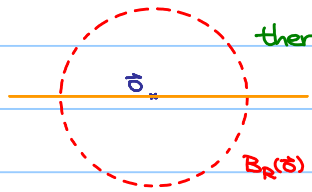
$S$ is said to be bounded if there exists $R > 0$ such that $S \subseteq B_R(\vec{0})$, in other words, $|\vec{x}| < R$ for all $\vec{x} \in S$. $S$ is said to be unbounded if $S$ is not bounded

Example 4.2

In $\mathbb{R}^2$,




$S$ is bounded




No matter how large $R$ is, there are always points outside $B_R(\vec{0})$.

$S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ is unbounded

Definition 4.6

Let $S \subseteq \mathbb{R}^n$.

$S$ is said to be compact if $S$ is closed and bounded

Exercise 4.3

Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere centered at the origin in $\mathbb{R}^3$.

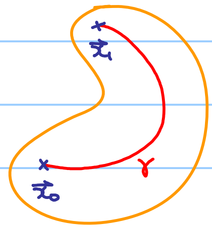
Show that $S^2$ is compact.

## Definition 4.7

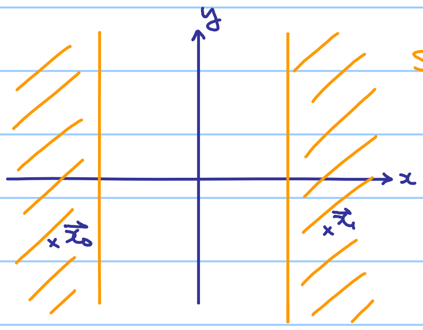
Let $S \subseteq \mathbb{R}^n$.

$S$ is said to be path connected if any two points in $S$ are connected by a curve in $S$ i.e. for all $\vec{x}_0, \vec{x}_1 \in S$, there exists a continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = \vec{x}_0$ and $\gamma(1) = \vec{x}_1$. $S$ is said to be disconnected if $S$ is not connected.

## Example 4.3




S is connected

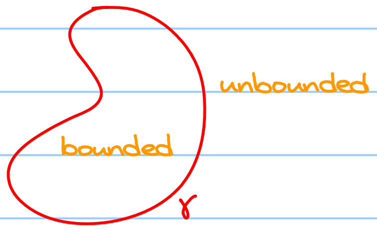



S is disconnected

## Theorem 4.1 (Jordan Curve Theorem)

A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two connected components, with one bounded and one unbounded.

## Example 4.4





Remark: An open connected subset in $\mathbb{R}^n$ is path connected.

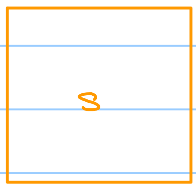
## Definition 4.8

Let $S \subseteq \mathbb{R}^n$.

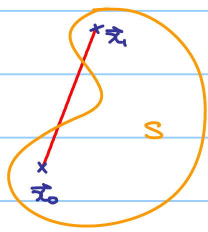
$S$ is said to be convex if for all $\vec{x}_0, \vec{x}_1 \in S$, the line segment joining $\vec{x}_0$ and $\vec{x}_1$ is entirely in $S$, i.e. $(1-t)\vec{x}_0 + t\vec{x}_1 \in S$ for all $t \in [0,1]$.

## Example 4.5

In $\mathbb{R}^2$,




S is convex




S is non-convex.

## § 5 Functions of Several Variables

### Real Valued Functions

**Definition 5.1**

Let $f: D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$.

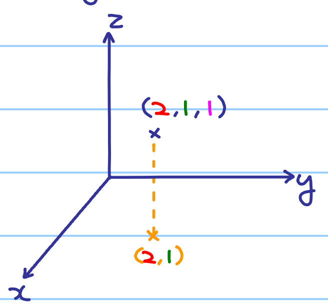
Then $f$ is said to be a real valued function (i.e output a real number) of $n$ variables and $D$ is the domain of the function $f$.

**Example 5.1**

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $z = f(x,y) = x + 2y - 3$.

$z = f(2,1) = 2 + 2(1) - 3 = 1$

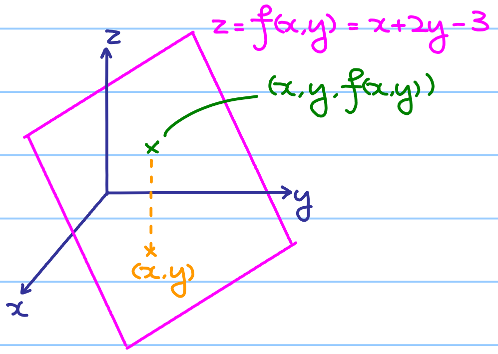
Given a point on xy-plane, the function returns the height.




Perform the above for every point on xy-plane and hence we obtain the graph of the function $z = f(x,y) = x + 2y - 3$.

$z = f(x,y) = x + 2y - 3 \Rightarrow x + 2y - z - 3 = 0$

and so the graph of $f$ is a plane.



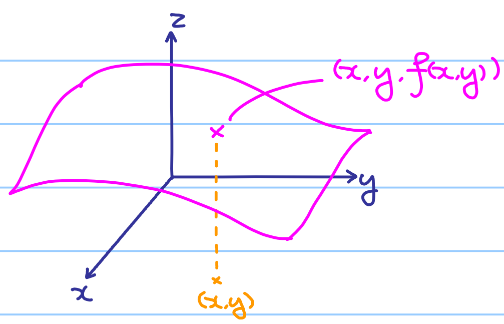

Think: What is the graph of a real valued function of two variables?

$f: D \to \mathbb{R}$ and $D \subseteq \mathbb{R}^2$

Given a point $(x,y) \in D$

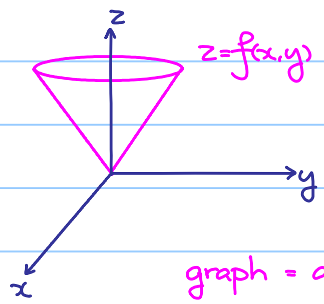
and associate a point $(x,y,f(x,y)) \in \mathbb{R}^3$.

The collection of all points associated as above is the graph of the function $f$.



Example 5.2

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $z = f(x,y) = \sqrt{x^2+y^2}$

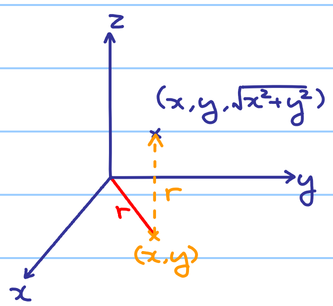



graph = a cone!

Why such a graph?

Recall: If $r = \sqrt{x^2+y^2}$ = distance between $(x,y)$ and $(0,0)$

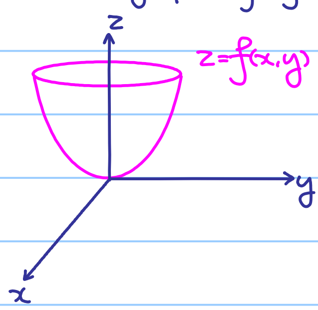
then $z = \sqrt{x^2+y^2} = r$




Exercise 5.1

What is the graph of $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $z = f(x,y) = x^2+y^2$ ?

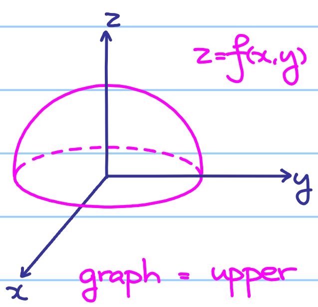
Ans:




Example 5.3

Let $D = \{(x,y) : x^2+y^2 \leq 1\}$ be the unit disk.

and let $f: D \to \mathbb{R}$ be a function defined by $z = f(x,y) = \sqrt{1-x^2-y^2}$




graph = upper hemisphere!

Why such a graph?

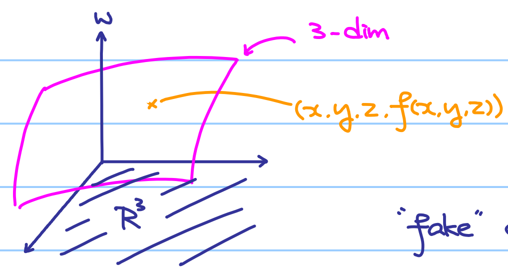
$z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 1$ and $z \geq 0$

Example 5.4

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$w = f(x,y,z) = x^2+y^2+z^2$.

Unfortunately, we do not have enough dimension to visualize the graph.




"fake" diagram.

Example 5.5

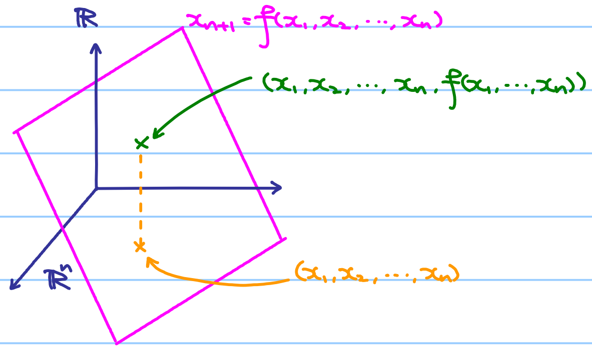
A function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be a linear function if

$f(x_1,x_2,\cdots,x_n) = A_1x_1 + A_2x_2 + \cdots + A_nx_n + B$ for some $A_1, A_2, \cdots, A_n, B \in \mathbb{R}$.

We can also write $f(\vec{x}) = \vec{a}\cdot\vec{x} + B$, where $\vec{a} = (A_1, A_2, \cdots, A_n)$, $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.

If we write $x_{n+1} = f(x_1,x_2,\cdots,x_n)$, then $A_1x_1 + A_2x_2 + \cdots + A_nx_n - x_{n+1} + B = 0$.

Therefore, the graph of $x_{n+1} = f(x_1,x_2,\cdots,x_n)$ defines an affine hyperplane in $\mathbb{R}^{n+1}$ with $(A_1, A_2, \cdots, A_n, -1)$ as a normal vector.




In particular, if $f(\vec{x}) = \vec{a}\cdot\vec{x}$ (i.e. $B=0$), $f$ is said to be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$.

Exercise: Show that (i) $f(\vec{x}_1+\vec{x}_2) = f(\vec{x}_1) + f(\vec{x}_2)$ for all $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$;

(ii) $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

## Vector Valued Functions

Definition 5.2

Let $f: D \to \mathbb{R}^m$ be a function, where $D \subseteq \mathbb{R}^n$

Then $f$ is said to be a vector valued function (i.e. output a vector in $\mathbb{R}^m$) of $n$ variables and $D$ is the domain of the function $f$.

Example 5.6

Let $f:\mathbb{R}^3 \to \mathbb{R}^2$ be a function defined by $(u,v) = f(x,y,z) = (x+y+z, xyz)$.

Then $f(1,2,4) = (7,8)$.

Remark: In this case, $u = x+y+z$ and $v = xyz$,

so we can also write $(u,v) = (u(x,y,z), v(x,y,z))$

where $u(x,y,z)$ and $v(x,y,z)$ are real valued functions from $\mathbb{R}^3$ to $\mathbb{R}$.

Example 5.7

Let $f:\mathbb{R}^3\to\mathbb{R}^2$ be a function defined by

$(u,v)=f(x,y,z)=(f_1(x,y,z),f_2(x,y,z))=(2x+3y+4z+5,\ 3x+2y+6z+1)$

Then, both $f_1(x,y,z)=2x+3y+4z+5$ and $f_2(x,y,z)=3x+2y+6z+1$ are linear functions.

By writing $(x,y,z)\in\mathbb{R}^3$ as a matrix in $M_{3\times1}(\mathbb{R})$, there is a simpler way to write:

$$f(x,y,z)=\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}f_1(x,y,z)\\f_2(x,y,z)\end{bmatrix}=\underbrace{\begin{bmatrix}2&3&4\\3&2&6\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\vec{x}}+\underbrace{\begin{bmatrix}5\\1\end{bmatrix}}_{\vec{b}}=A\vec{x}+\vec{b}$$

, where $A\in M_{2\times3}(\mathbb{R})$, $\vec{x}\in M_{3\times1}(\mathbb{R})$, $\vec{b}\in M_{2\times1}(\mathbb{R})$.

Definition 5.3

Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a function.

$f$ is said to be a linear function if $f(\vec{x})=A\vec{x}+\vec{b}$, where $A\in M_{m\times n}(\mathbb{R})$, $\vec{b}\in M_{m\times1}(\mathbb{R})$

In particular, if $\vec{b}=\vec{0}$, then $f$ is said to be a linear transformation.

Exercise 5.1

Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a function.

Show that $f$ is a linear transformation if and only if $f$ satisfies

(i) $f(\vec{x}_1+\vec{x}_2)=f(\vec{x}_1)+f(\vec{x}_2)$ for all $\vec{x}_1,\vec{x}_2\in\mathbb{R}^n$;

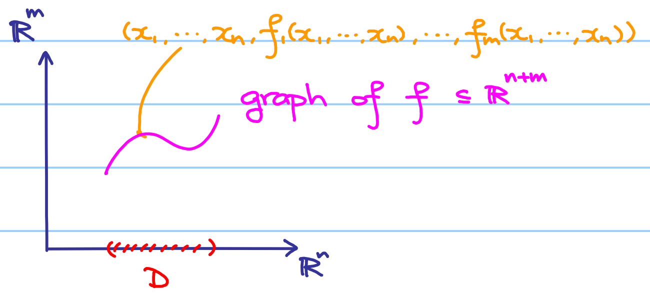
(ii) $f(c\vec{x})=cf(\vec{x})$ for all $\vec{x}\in\mathbb{R}^n$ and $c\in\mathbb{R}$.

In general, let $D\subseteq\mathbb{R}^n$, we may have a function $f:D\to\mathbb{R}^m$,

(i.e. we input a point $(x_1,x_2,\cdots,x_n)\in D\subseteq\mathbb{R}^n$ into $f$, it returns a point in $\mathbb{R}^m$)

we may write $f(x_1,x_2,\cdots,x_n)=(f_1(x_1,x_2,\cdots,x_n),f_2(x_1,x_2,\cdots,x_n),\cdots,f_m(x_1,x_2,\cdots,x_n))$,

where each $f_i:D\to\mathbb{R}$ is a real valued function.



## Level Sets

Question: How do we understand a real valued function in more detail?

Definition 5.3

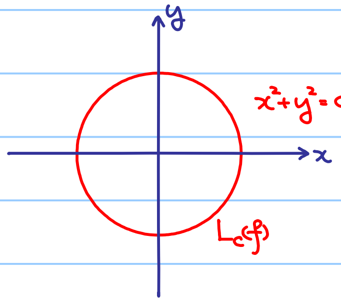
Let $f: D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$.

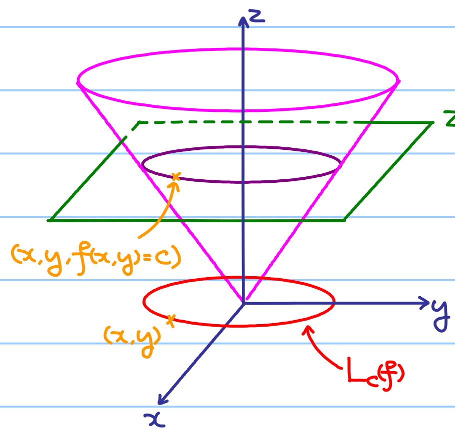
Let $c \in \mathbb{R}$. The set $L_c(f) = f^{-1}(c) = \{(x_1, x_2, \dots, x_n) \in D : f(x_1, x_2, \dots, x_n) = c\}$ is said to be the level set of $f$ corresponding to $c$.

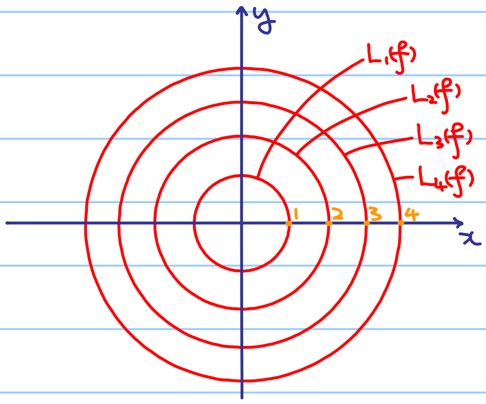
Example 5.7

Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $z = f(x,y) = \sqrt{x^2+y^2}$

Let $c>0$, $L_c(f) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = \sqrt{x^2+y^2} = c\} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c^2\}$



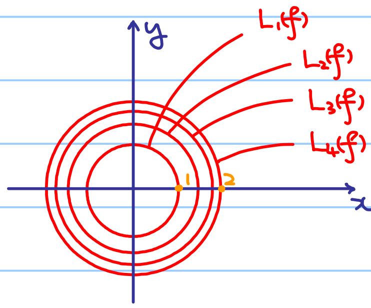

If $(x,y) \in L_c(f)$, then $f(x,y) = c$ and so $(x,y,f(x,y)) = (x,y,c)$ is a point lying on the intersection of the graph of $f$ and the plane $z=c$.

Remark: $L_0(f) = \{(0,0)\}$ and $L_c(f)$ is the empty set if $c<0$.

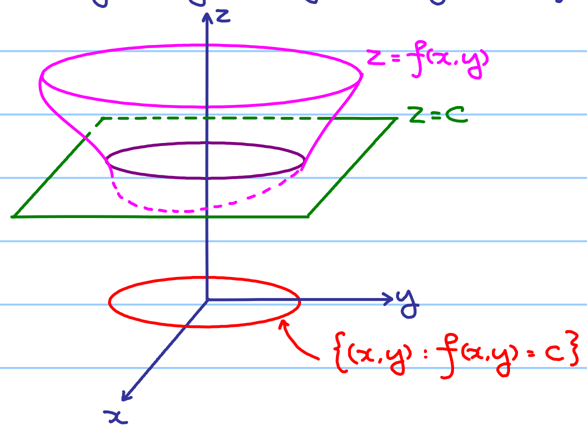
Example 5.8

Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $z = f(x,y) = x^2+y^2$

Let $c>0$, $L_c(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c\}$



In general, a curve in $\mathbb{R}^2$ may be given by an equation $f(x,y)=c$.



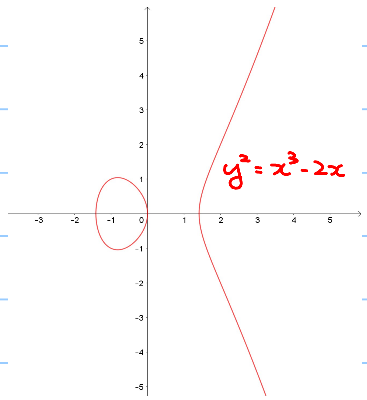

Solutions of $f(x,y)=c$ form a curve.

Example 5.9

Consider the equation $y^2 = x^3 - 2x$.

In fact, we can move all the terms on the right hand side to the left, so $y^2 - x^3 + 2x = 0$.

Let $f(x,y) = y^2 - x^3 + 2x$. Then, solutions of $y^2 = x^3 - 2x$ are exactly solutions of $f(x,y) = 0$ which form a curve in $\mathbb{R}^2$.




Example 5.10

Let $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $w = f(x,y,z) = x^2 + y^2 + z^2$

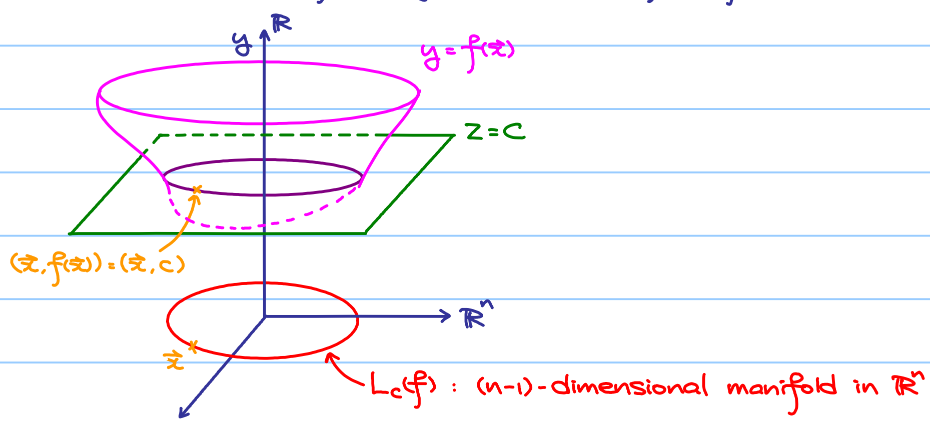
If $c > 0$, $L_c(f) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = c\}$ sphere centered at the origin with radius $\sqrt{c}$.

Remark: $L_0(f) = \{(0,0)\}$ and $L_c(f)$ is the empty set if $c < 0$.

Idea: Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$, and let $c \in \mathbb{R}$.

(With certain assumption on $f$)

$L_c(f) = f^{-1}(c) = \{(x_1, x_2, \dots, x_n) \in D : f(x_1, x_2, \dots, x_n) = c\}$ defines a $(n-1)$-dimensional manifold (generalization of surface) in $\mathbb{R}^n$



Example 5.11

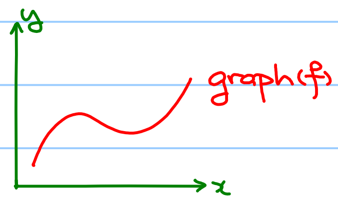
Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then, $\text{graph}(f) = \{(x,y) : y=f(x),\ x\in\mathbb{R}\} \subseteq \mathbb{R}^2$

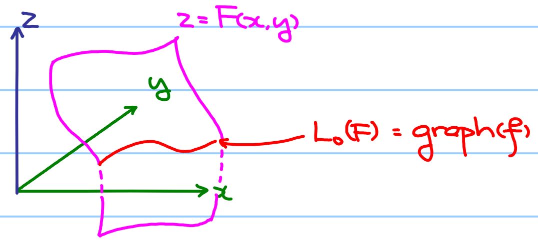
However, the graph of $f$ can also be regarded as a level set of some function as the following:

Note that $y=f(x) \Rightarrow f(x)-y=0$

Let $F:\mathbb{R}^2\to\mathbb{R}$ be a function defined by $z=F(x,y)=f(x)-y$.

Then $F(x,y)=0 \Leftrightarrow y=f(x)$ and so $L_0(F)=\{(x,y)\in\mathbb{R}^2 : F(x,y)=0\} = \text{graph}(f)$.




Think: If $f:\mathbb{R}^n\to\mathbb{R}$ be a function.

How do we express graph(f) as a level set of some function?

Before Moving on

Idea: In the following discussion on limits, continuity and differentiability of functions, we will separate the discussion into cases:

0) $f:\mathbb{R}\to\mathbb{R}$ (done!)

1) $f:\mathbb{R}^n\to\mathbb{R}$

2) $f:\mathbb{R}^n\to\mathbb{R}^m$

Generalization!